

QUANTUM PROCESSES IN RESONATORS WITH MOVING WALLS

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Abstract.

The behavior of the electromagnetic field in an ideal cavity with oscillating boundary is considered in the resonance long-time limit. The rates of photons creation from vacuum and thermal states are evaluated. The squeezing coefficients for the field modes are found, as well as the backward reaction of the field on the vibrating wall.

1. Field Quantization in a Cavity of Variable Length

Here we give the results of our recent investigations relating to the behavior of the quantized modes of the electromagnetic field inside a resonator with oscillating walls. We consider the electromagnetic field in an empty resonator formed by two ideal conducting plain boundaries $x=0$ and $x=L(t)$, and restrict ourselves to the linearly polarized modes with the electric vector parallel to the boundaries. Then the field can be described by means of the single scalar equation for the corresponding component of the vector potential with the nonstationary boundary conditions [1] (we assume $c=1$)

$$\varphi_{tt} - \varphi_{xx} = 0, \quad 0 < x < L(t); \quad \varphi(0,t) = \varphi(L(t),t) = 0 \quad (1)$$

The quantization procedure in this case was proposed by Moore [1]. (Another approach including the case of a massive boson scalar field was investigated in ref.[2].) The starting point of Moore's method is the following choice of the fundamental solutions of eq.(1),

$$w_{\pm}(x,t) = (4\pi n)^{-1/2} \left\{ \exp[-i\pi n R(t-x)] - \exp[-i\pi n R(t+x)] \right\}, \quad (2)$$

function $R(\xi)$ being a solution of the functional equation

$$R[t+L(t)] - R[t-L(t)] = 2 \quad (3)$$

In the stationary case $L(t) = L_0$ the solution of eq. (3) is trivial: $R^{(0)}(\xi) = \xi/L_0$. Thus mode functions are usual standing waves

$$\psi_n^{(0)}(x, t) = i(\pi n)^{-1/2} \sin(\pi n x/L_0) \exp(-i\pi n t/L_0). \quad (4)$$

An approximate solution of (3) for a slowly moving wall was found in [1]. But in the most interesting case of the parametric resonance

$$L(t) = L_0 [1 + \varepsilon \sin(\omega_q t)], \quad \omega_q = \pi q/L_0, \quad q = 1, 2, \dots, |\varepsilon| \ll 1 \quad (5)$$

that solution appears valid only for not very large values of time satisfying the restriction $\varepsilon \omega t/L_0 \ll 1$. The correct asymptotic expression for the function $\rho(t) = R(t) - t$ in the long-time limit $\varepsilon \omega t \gg 1$ was obtained in refs. [3-5] ($L_0 = c = 1$, $\xi = \exp[(-1)^{q+1} \pi q \varepsilon t]$):

$$\rho(t) = - (2/\pi q) \cdot \text{Im} \left\{ \ln [1 + \xi + \exp(i\pi q t) (1 - \xi)] \right\}, \quad (6)$$

For the motionless walls the field operator $\hat{\phi}$ in the Heisenberg picture can be developed over the set of functions $\psi_n^{(0)}(x, t)$:

$$\hat{\phi}(x, t) = \sum_n \left\{ \hat{b}_n \psi_n^{(0)}(x, t) + \hat{b}_n^+ [\psi_n^{(0)}(x, t)]^* \right\}, \quad [\hat{b}_n, \hat{b}_m^+] = \delta_{nm}. \quad (7)$$

If the right wall oscillates only during the time interval $0 \leq t \leq T$, then for $t > T$ the field operator can be written in two forms,

$$\hat{\phi} = \sum_n \left\{ \hat{b}_n \psi_n + \hat{b}_n^+ \psi_n^* \right\} = \sum_m \left\{ \hat{a}_m \psi_m^{(0)} + \hat{a}_m^+ [\psi_m^{(0)}]^* \right\}. \quad (8)$$

where $\psi_n(x, t)$ is the solution of the nonstationary problem (1) coinciding with $\psi_n^{(0)}$ at $t < 0$. It seems reasonable to assume that measuring devices react to steady-state standing waves (4) which are wave functions of physical quantum states possessing definite energy values. Then just the set of operators (\hat{a}, \hat{a}^+) has the physical sense at $t > T$. Since all quantum properties of the field were defined with respect to the state determined by the set of operators (\hat{b}, \hat{b}^+) (which were "physical" operators for $t < 0$), we have to expand the "new" operators (\hat{a}, \hat{a}^+) over the "old" ones (\hat{b}, \hat{b}^+) ,

$$\hat{a}_m = \sum_n \left\{ \hat{b}_n \alpha_{nm} + \hat{b}_n^+ \beta_{nm}^* \right\}. \quad (9)$$

To calculate the Bogoliubov coefficients α_{nm} and β_{nm} one should take into account that both systems of mode functions (2) and (4) consti-

complete orthonormal sets with respect to the scalar product [1]

$$L(t) \\ (\psi, \chi) = -i \int_0^L dx \left\{ \psi \chi_t^* - \psi_t \chi^* \right\}, \quad (\chi_t \equiv \partial \chi / \partial t) \quad (10)$$

Then the following relations can be obtained [3-5],

$$\left. \begin{array}{l} \alpha_{nm} \\ \beta_{nm} \end{array} \right\} = \frac{1}{2} (m/n)^{1/2} \int_{t/L_0-1}^{t/L_0+1} dx \cdot \exp \left\{ -i\pi [n\rho(L_0 x) + (n \mp mx)] \right\}, \quad (11)$$

The detailed calculations of these integrals were performed in [3-5].

The final result for $q = 2r$ is as follows ($\delta = \sqrt{\epsilon}/\pi r$),

$$\left. \begin{array}{l} \beta_{nm} \\ \alpha_{nm} \end{array} \right\} = \sqrt{\frac{m}{n}} \frac{\sin[\pi(2rn\delta \pm m)/2r]}{\pi(2rn\delta \pm m)} \cdot \frac{\sin[\pi(n \pm m)]}{\sin[\pi(n \pm m)/2r]} \exp[i\pi(n \pm m)(1 - 1/2r)]. \quad (12)$$

In the main resonance case of $r=1$ the following expression for the moduli squared of the Bogoliubov coefficients can be obtained,

$$\left. \begin{array}{l} |\beta_{nm}|^2 \\ |\alpha_{nm}|^2 \end{array} \right\} = \frac{m}{n\pi^2} \frac{[1 - (-1)^m \cos(2n\pi\delta)]}{(2n\delta \pm m)^2} [1 + (-1)^{m+n}]. \quad (13)$$

2. Rates of Photons Generation

The total number of photons created in the m -th mode from the vacuum state to the time instant t equals

$$P_m = \langle 0 | \hat{a}_m^+ \hat{a}_m | 0 \rangle = \sum_n |\beta_{nm}|^2. \quad (14)$$

Omitting the details of calculations given in [3-5] we present the final result ($r = 1$)

$$P_m \approx (m\pi^2)^{-1} [\ln(m/2\delta) - (-1)^m \ln(1/2\pi\delta)]. \quad (15)$$

Since in the case under study $\delta(t) = \exp(-\pi\epsilon t)/\pi$, we get the following rate of photons generation in the m -th mode when the wall vibrates at the twice frequency of the first resonator eigenmode for $\epsilon \gg 1$:

$$dP_m/dt = (\epsilon/\pi m) [1 - (-1)^m]. \quad (16)$$

This result is valid in fact only for not very large numbers m . Since in real situations we should limit time t by the resonator relaxation time τ (due to the dissipation inside walls), the maximum number of photons generated in the m -th mode equals approximately

$$P_m^{\max} \approx (\pi^2 m)^{-1} \left\{ 2\epsilon Q(m)/m + O(\ln(m+1)) \right\}, \quad (17)$$

where $Q(m)$ is the quality factor of the resonator's m -th mode.

Formulas (15)-(17) essentially differ from the results of refs. [5,7], where the problem of photons creation in a resonator with oscillating ideal mirror was also considered. However the authors of that papers did not take into account the deep reconstruction of the field modes inside the resonator in the long-time limit. Therefore the rate of photons generation obtained in [6] and [7] was proportional in essence to $(\epsilon\omega_q)^2$, whereas our formulas show that this rate is proportional to the first power of the product $\epsilon\omega_q$. The quadratic law $P_q \propto (\epsilon\omega_q)^2$ is valid only in the short-time approximation under the condition $P_q \ll 1$, as was shown in [8].

If the initial density matrix of the field corresponds to the Planck distribution with finite temperature, then the average number of additional "thermal" quanta created in the m -th mode equals [4]

$$\begin{aligned} \Delta P_m &= P_m - P_m^{\text{vac}} = \sum_k \left\{ |\alpha_{km}|^2 + |\beta_{km}|^2 \right\} \cdot \left\{ \exp(k/\Theta) - 1 \right\}^{-1} \\ &= 2(\pi^2 m)^{-1} \left[1 - (-1)^m \right] \sum_{j=1}^{\infty} \ln \left\{ \frac{\exp(j/\Theta) + 1}{\exp(j/\Theta) - 1} \right\} + O(2\delta/m), \end{aligned} \quad (18)$$

where $\Theta = \kappa T L_0 / \pi \hbar c$, κ is Boltzmann's constant, T - temperature.

The final number of "thermal" photons does not depend on time. Moreover, in the even modes it is almost zero up to the terms of the order of $2\delta/m$. In the low temperature limit $\Theta \ll 1$ and for $\epsilon \gg 1$ [4]

$$\Delta P_m = 4(\pi^2 m)^{-1} \left[1 - (-1)^m \right] \cdot e^{-1/\Theta} \ll P_m^{\text{vac}}. \quad (19)$$

In the high temperature limit one gets [4]

$$\Delta P_m = (\Theta/2m) \left[1 - (-1)^m \right] + O(\ln(2\Theta)). \quad (20)$$

If the resonator has a finite quality factor $Q(m)$ in the m -th mode, then the temperature corrections can be neglected provided

$$\Theta \ll 4\epsilon Q(m) / (\pi^2 m); \quad \epsilon Q \gg 1. \quad (21)$$

3. Squeezing Coefficients

Now let us consider variances of canonical coordinates and momenta operators (quadrature components)

$$\hat{x}_m = (\hat{a}_m + \hat{a}_m^+)/\sqrt{2}, \quad \hat{p}_m = i(\hat{a}_m^* - \hat{a}_m)/\sqrt{2}. \quad (22)$$

If the initial quantum state of the field was vacuum or coherent one, then the following general formulas are valid [8],

$$\sigma_{x_m x_m} = \frac{1}{2} + \sum_n \left\{ |\beta_{nm}|^2 + \operatorname{Re}(\alpha_{nm} \cdot \beta_{nm}^*) \right\}, \quad (23)$$

$$\sigma_{p_m p_m} = \frac{1}{2} + \sum_n \left\{ |\beta_{nm}|^2 - \operatorname{Re}(\alpha_{nm} \cdot \beta_{nm}^*) \right\}, \quad (24)$$

$$\sigma_{x_m p_m} = \sum_n \operatorname{Im}(\alpha_{nm} \cdot \beta_{nm}^*). \quad (25)$$

In the case of $q=2$ we have [4,5]

$$\sigma_{x_m x_m}(t \rightarrow \infty) = \frac{1}{2} - (\pi^2 m)^{-1} \left\{ 1 - (-1)^m - (\pi m) \cdot \operatorname{si}(\pi m) \right\}, \quad (26)$$

where $\operatorname{si}(x)$ means the integral sine function:

$$\operatorname{si}(x) = \int_{-\infty}^x dt \cdot \sin(t)/t. \quad (27)$$

We see that the variance is always less than its value in the vacuum state $\sigma_{\text{vac}} = 1/2$. This means that the field occurs in the squeezed state. The relative squeezing coefficient $K_m = 1 - 2\sigma_{x_m x_m}$ assumes the maximum value $K_1 = 0.22$ for $m = 1$. For large $m \gg 1$ this coefficient slowly decreases according to the asymptotic formula

$$K_m \approx 2/(\pi^2 m). \quad (28)$$

The canonical momentum variance increases in time according to the same law as the number of created quanta (15). The general dependences of variances on time are rather intricate. As was shown in [8], in the short time limit $\pi t \ll 1$ there is a small squeezing in the canonical momentum variance: $\sigma_{pp} \approx \frac{1}{2}(1 - \pi \pi t)$ (for $m = 1$). Meanwhile in the long time limit the situation is quite opposite: there is some squeezing of the canonical coordinate, and unlimitedly growing in time variance of the canonical momentum. As to the covariance of the coordinate and momentum (25), it turns out to be equal to zero up to the terms of the order of $(\pi t)^{-1}$. This means that the field occurs in a squeezed but uncorrelated state. Nonetheless, this state is not a

minimum uncertainty state, since $\sigma_{pp}\sigma_{xx} \gg 1$ when $z \gg 1$. This is explained by a strong intermode interaction.

4. Back Reaction on the Oscillating Wall from the Field

It is well known that vacuum fluctuations of electromagnetic field result in an attractive Casimir's force between uncharged conducting plates [9-11]. The general expression for the force pressing the moving wall (more precisely the T_{11} -component of the energy-momentum tensor of the field) was calculated in [10,12]:

$$F = - [g(L-L(t)) + g(L+L(t))], \quad (29)$$

where function $g(y)$ is expressed through R -function introduced according to eqs. (2) and (3) as follows (in dimensionless units; remind that we consider the case of "one-dimensional" electrodynamics),

$$g(y) = \frac{1}{24\pi} \left\{ \frac{R'''(y)}{R'(y)} - \frac{3}{2} \left[\frac{R''(y)}{R'(y)} \right]^2 + \frac{\pi^2}{2} [R'(y)]^2 \right\}. \quad (30)$$

In the case of motionless wall (29) and (30) lead to the known expression for Casimir's force in one dimension

$$F^{(0)} = -\pi\hbar c/24L_0^2 \quad (31)$$

The corrections to (31) in the limit of small velocities of the wall (with respect to the velocity of light) were calculated in [13]. The additional force appears attractive and proportional to the square of wall's velocity. Here we calculate the same force using the long-time asymptotics of R -function (6). Since $|d\xi/dt| \approx |\xi| \ll \xi$, we can differentiate R -function with respect to time believing parameter ξ to be constant. Then the first three derivatives are as follows,

$$R'(L) = 2\xi\psi(L), \quad (32)$$

$$R''(L) = 2\xi(1-\xi^2)\pi q \sin(\pi q L) \psi^2(L), \quad (33)$$

$$R'''(L) = 2\xi(1-\xi^2)(\pi q)^2 \left[(1+\xi^2)\cos(\pi q L) + (1-\xi^2)[1+\sin^2(\pi q L)] \right] \psi^3(L), \quad (34)$$

$$\psi(L) = [1 + \xi^2 + (1-\xi^2)\cos(\pi q L)]^{-1} \quad (35)$$

Since the force exhibits rapid oscillations, it seems reasonable to average all time dependent functions contained in (35) over the period of oscillations $T = 2/q$. All integrals can be calculated exactly with the aid of formula ([14], eq.2.5.16(22))

$$\int_0^{\pi} \frac{\cos(n\alpha)}{a + b\cos(\alpha)} d\alpha = \frac{\pi}{(a^2 - b^2)^{1/2}} \left[\frac{(a^2 - b^2)^{1/2} - a}{b} \right]^n \quad (36)$$

so that we have

$$\langle R'^2 \rangle = \frac{1}{2} \langle \xi + \xi^{-1} \rangle, \quad (37)$$

$$\langle R'' R'^2 \rangle = \langle R'' R' \rangle = \frac{1}{2} (\pi q)^2 \langle \xi + \xi^{-1} - 2 \rangle, \quad (38)$$

Inserting these expressions into (29) and (30) we get finally in dimensionless units

$$\langle F \rangle = - \frac{\pi}{24} [q^2 + \frac{1}{2}(1 - q^2)(\xi + \xi^{-1})] \quad (39)$$

For $q = 1$ this formula coincides with (31). Note that this is not the resonance case (the minimal resonance value is $q = 2$), so that photons are not created inside the resonator, and the force conserves its vacuum value. For $q \geq 2$ we have not attraction, but an exponentially increasing pressure on the oscillating wall due to the creation of real photons in the cavity. By the way, formula (39) shows distinctly that for $t \rightarrow \infty$ the physical results do not depend on the sign of the parameter ϵ characterizing the dimensionless amplitude of wall's vibrations, since $\langle F \rangle$ is proportional to $\exp(|\epsilon| \pi q t)$.

5. Discussion.

Let us summarize the main results. We have presented a new solution for mode functions of the electromagnetic field inside an ideal cavity with oscillating wall in the long-time resonance limit. It appears that the field modes structure is significantly changed in this limit in comparison with the case of motionless boundaries. It is seen distinctly if one compares, e.g., the time derivatives of functions $R^{(0)}(\xi)$ and $R(\xi)$ given by (6): in the motionless case one gets unity (in dimensionless units), whereas in the long-time resonance limit the corresponding value appears much less than unity for almost all instants of time excepting those when $\cos(\pi q t)$ is very close to ± 1 (see eq. (32)). Physically this change of the field modes structure manifests itself in the transition from the quadratic law of photons generation in the short-time approximation to the linear law in the long-time asymptotics. We have established also the possi-

bility of obtaining some squeezing (although rather moderate) in the resonance modes.

REFERENCES

- [1]. G.T.Moore, J. Math. Phys. **11**, 2979 (1970).
- [2]. M.Razavy, Lett. Nuovo Cim. **37**, 449 (1983); M.Razavy J.Terning, Lett. Nuovo Cim. **41**, 561 (1984); Phys. Rev. **D31**, 307 (1985).
- [3]. V.V.Dodonov, A.B.Klimov, D.E.Nikonov, in Proceedings of a Workshop on Squeezed States and Uncertainty Relations, College Park, March 1991, edited by D.Han, Y.S.Kim, and W.W.Zachary (NASA Conference Publication 3135, 1992), p.199-215.
- [4]. V.V.Dodonov, A.B.Klimov, Journal of Soviet Laser Research (Plenum Publishing, London, N.Y.) **13**, 230 (1992).
- [5]. V.V.Dodonov, A.B.Klimov, Phys. Lett. A **167**, 309 (1992).
- [6]. S.Barkar, J. Phys. **A21**, 971 (1988); in: *Photons and Quantum Fluctuations*, edited by E.R.Pike and H.Walther (Adam Hilger, Bristol, 1989), p.151.
- [7]. L.A.Rivlin, Kvantovaya Elektronika (Sov.Phys.-Quant.Electron.) **6**, 2248 (1979).
- [8]. V.V.Dodonov, A.B.Klimov, V.I.Man'ko, Phys. Lett. **A149**, 225 (1990); J. Sov. Laser Research **12**, 439 (1991).
- [9]. H.B.S.Casimir, Proc. Kon. Ned.Acad. Wetenskap, **51**, 793 (1948).
- [10]. N.Birrell and P.Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge, England, 1982).
- [11]. V.M.Mostepanenko, N.N.Trunov, Uspekhi Fiz. Nauk (Sov. Fiz. - Uspekhi) **156**, (1988).
- [12]. S.A.Fulling and P.C.W.Davies, Proc.Roy.Soc.(London) **348**, 393 (1976).
- [13]. V.V.Dodonov, A.B.Klimov, V.I.Man'ko, Phys.Lett.**A142**, 511 (1989).
- [14]. A.F.Prudnikov, Yu.A.Brychkov, O.I.Marichev, *Integrals and Sums* (Nauka, Moscow, 1981).